Pergamon

Int. J. Multiphase Flow Vol. 22, No. 2, pp. 259–272, 1996

0301-9322(95)00079-8
0301-9322/96 $15.00 + 0.00

THE HEAT AND MASS TRANSFER OF A VAPOR BUBBLE WITH TRANSLATORY MOTION AT HIGH NUSSELT NUMBERS

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(Received 15 February 1995; in revised form 22 November 1995)

Abstract—Within the framework of the assumptions that the processes of condensation or evaporation in subcooled or superheated liquids are controlled by convective heat transfer, and that the flow past a spherical bubble of a changeable radius is both potential and axisymmetric, the heat exchange of a vapor bubble at high Nusselt numbers is considered. Asymptotic expressions for the Nusselt number, taking into account the unsteady character of the thin thermal boundary layer due to the changeability of the bubble radius and the relative velocity, are obtained. Calculations and comparisons between the present theory and previous numerical and experimental data are performed.

Key Words: bubble dynamics, vapor bubble, condensation, evaporation, heat and mass transfer, subcooled and superheated liquids, thermal boundary layer, physico-chemical hydrodynamics.

1. INTRODUCTION

The dynamics of a vapor bubble in subcooled or superheated liquids have been investigated by various authors since the pioneering work of Rayleigh (1917). The Rayleigh regime of bubble collapse controlled by the inertia of the liquid is not typical and often the collapse process is limited by the ability of the liquid to conduct the heat of condensation (the so-called thermal regime). The criterion, showing which of these two mechanisms is limiting the process at moderate or high Jacob numbers and in the absence of translatory motion, was introduced by Florschuetz & Chao (1965).

Most theoretical works deal with the thermal regime of bubble dynamics without translatory motion (for example, Plesset & Zwick 1952; Plesset & Zwick 1954; Scriven 1959; Zuber 1961; Florschuetz & Chao 1965; Prosperetti & Plesset 1978; Nigmatulin *et al.* 1981; Korabelnikov *et al.* 1981; Zuong Ngoc Hai & Khabeev 1983; Nakoryakov *et al.* 1983; Nigmatulin 1987; Okhotsimskii 1988; Gumerov 1989; Nigmatulin *et al.* 1991; etc.). As a rule, translatory motion caused by gravitation and acceleration of the flow takes place. The flow past the bubble can heavily influence the thermal regime of bubble collapse due to the convection and consequent increase of the condensation rate.

An expression for the total mass flux on a spherical gas bubble in steady motion with a thin diffusion boundary layer can be found in the monograph of Levich (1959). This formula is also applicable to calculate the quasi-steady heat flux on vapor bubbles, because of the analogy between the heat and mass diffusion and identical mathematical models.

Experimental and numerical studies of vapor bubble collapse with translatory motion were carried out by Wittke & Chao (1967). There, the sphericity of the bubble, constant velocity of the flow far from the bubble and axisymmetric temperature and velocity profiles are assumed in the mathematical model; results of the calculations are consistent with experiments. The same assumptions were later used by Sagitov & Khabeev (1989) in their numerical simulations. However, they also took into account the inertia of the liquid at radial motion and the vapor elasticity that lead to possible radial oscillations. Further development of the model, including the conservation of momentum equation for spherical bubbles, and corresponding simulations were performed in the work of Zolovkin *et al.* (1994).

In the paper of Florschuetz *et al.* (1969), extensive experimental data are presented for growth

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rates of vapor bubbles in various superheated liquids under normal and zero gravity conditions. The authors found that the initial stage of bubble growth can be described properly by using the Scriven exact self-similar solution (Scriven 1959). Some influence of translatory motion was observed in the next stages of bubble growth under normal gravitation. A review of previous works can also be found in this article.

The work of Ruckenstein & Davis (1971) is remarkable because an exact solution of the convective heat conduction equation describing the heat transfer in the unsteady thin boundary layer near a spherical bubble is obtained. Comparisons with known experimental data on bubble growth with translatory motion support analytical solutions. Unfortunately, the expression found for the total heat flux to the bubble is unwieldy and can hardly be used in complex models of multiphase flows with vapor bubbles.

A number of researchers have used some model simplifications and experimental data to obtain simple formulae describing bubble dynamics with translatory motion (Moalem & Sideman 1973; Voloshko *et al.* 1973; Akiyama 1973; Dimic 1977; Nordmann & Mayinger 1981; Chen 1985; Mayinger *et al.* 1991). The last three works consist of reviews in the field and comparisons of their experimental data and approximations with calculations using formulae of previous authors.

Reviews of analytical works dealing with heat and mass transfer near rigid particles, droplets and bubbles with translatory motion can also be found in monographs by Gupalo *et al.* (1985) and Dilman & Polyanin (1988). The quasi-stationary dependencies of Nusselt number on the flow parameters (Peclet, Reynolds and Weber numbers) were studied there. Also, effects such as chemical reactions and droplet and bubble deformations were taken into account in the cited works.

The idea of the present work came after the ICHMT seminar held in Dubrovnik, Yugoslavia in 1990, where Professor F. Mayinger presented some of the results of his and his co-workers' investigations on bubble collapse in subcooled liquids. These results demonstrate a substantial quantitative disagreement between existing simplified formulae for bubble dynamics and experiments.

The present work has two goals. First, to conduct an analytical study of the non-stationary heat and mass exchange between the moving bubble and surrounding liquid and to classify the possible regimes of bubble growth or collapse. Second, to obtain sufficiently simple formulae describing the heat and mass transfer near vapor bubbles that are both consistent with experiments and can be used for various descriptions of dynamic processes in multiphase flows such as cavitation, boiling, waves in vapor–liquid mixtures, etc. Selected results of the present work were presented at the Euromech Colloquium *Flows with Phase Transitions* held in Göttingen, Germany in 1995.

2. BASIC ASSUMPTIONS AND EQUATIONS

Usually we can accept the following assumptions (see, for example, Wittke & Chao 1967):

(1) the vapor bubble is a sphere of radius $a(t)$, which moves with the translational velocity $U(t)$ in an incompressible liquid having a constant temperature T_0, pressure p_0 and zero velocity at infinity;

(2) the profile of the velocities in the liquid ($\mathbf{v}_L$) is axisymmetric and potential ($\mathbf{v}_L = \nabla_r \Phi$);

(3) the pressure in the bubble is constant p_0;

(4) the local thermodynamic equilibrium between the liquid and the vapor holds and, thus, the temperature of the interface is equal to the saturation temperature $T_s(p_0)$;

(5) the thickness of the thermal boundary layer is small if compared with the bubble size;

(6) the vapor is a perfect gas and its density ρ_G is much less than the density of the liquid ρ_L;

(7) both the thermal (λ_L) and temperature (κ_L) conductivities of the liquid are constant.

Thus, the dynamics of the bubble mass m_G can be described by the equations

$$l\frac{dm_G}{dt} = -\int_S \mathbf{q}_L|_S \mathbf{n}\, dS,\quad \mathbf{q}_L = -\lambda_L \nabla_r T_L,\quad m_G = \frac{4}{3}\pi a^3 \rho_G \qquad [1]$$

where T_L and $\mathbf{q}_L$ are the temperature of the liquid and the heat flux, $\mathbf{n}$ is the outer normal vector to the sphere surface and l is the latent heat of evaporation.

To find the temperature distribution we must solve the following problem

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_L \nabla_r\right)T_L = \kappa_L \Delta_{rr} T_L,\ \ T_L\big|_{r=a(t)} = T_S,\ \ T_L\big|_{r=\infty} = T_0$$

$$\mathbf{v}_L = \nabla_r \phi_L,\ \ \phi_L = -\frac{a^2}{r}\frac{da}{dt} - U\left(r + \frac{a^3}{2r^2}\right)\cos\vartheta,\ \ \Delta_{rr} = \nabla_r \nabla_r. \qquad [2]$$

Here r and ϑ are the spherical polar coordinates connected with the center of the bubble.

If $U(t)$ is a known function and initial conditions are determined, then the system of [1] and [2] is closed, and the bubble radius dynamics $a(t)$ can be found.

3. DIMENSIONLESS PARAMETERS AND CHARACTERISTIC TIMES

Let us define t_* as the characteristic time of bubble collapse in a subcooled liquid. This time depends on the initial radius a_0, initial or characteristic velocity U_0 and the thermophysical properties of the phases. We can introduce the following dimensionless parameters and variables:

$$t' = \frac{t}{t_*},\ \ \eta = \frac{r}{a(t)},\ \ u = \frac{T_L - T_0}{T_S - T_0},\ \ a' = \frac{a}{a_0},\ \ U' = \frac{U}{U_0},\ \ \mathrm{Pe} = \frac{2aU}{\kappa_L},\ \ \mathrm{Nu} = -\int_0^\pi \frac{\partial u}{\partial \eta}|\eta = 1 \sin\vartheta\, d\vartheta$$

$$\mathrm{Ja} = \frac{\rho_L c_L (T_S - T_0)}{\rho_G l},\ \ \mathrm{Pe}_0 = \frac{2a_0 U_0}{\kappa_L},\ \ \mathrm{St} = \frac{2a_0}{U_0 t_*},\ \ \epsilon = \frac{(\kappa_L t_*)^{1/2}}{a_0},\ \ \delta = \frac{\mathrm{Ja}}{\mathrm{Pe}_0^{1/2}} \qquad [3]$$

where c_L is the specific heat of the liquid; and Pe, Ja, Nu and St are the Peclet, Jacob, Nusselt and Strouhal numbers, respectively.

Note, that the Jacob number introduced by [3] is negative if we have bubble growth (Ja < 0) and positive in the case of bubble collapse (Ja > 0). Below we consider the case of bubble collapse in subcooled liquids and in estimations of magnitudes we assume that Ja > 0. The estimations also hold for Ja < 0, but in this case we should replace Ja by $|\mathrm{Ja}|$.

Using variables [3] we can represent [1] and [2] in the form:

$$2a'\frac{da'}{dt'} = -\epsilon^2 \mathrm{Ja}\mathrm{Nu} \qquad [4]$$

$$(a')^2\frac{\partial u}{\partial t'} - a'\frac{da'}{dt'}\left(\eta - \frac{1}{\eta^2}\right)\frac{\partial u}{\partial \eta} + \frac{2a'U'}{\mathrm{St}}(\nabla_\eta \phi')\nabla_\eta u = \epsilon^2 \Delta_{\eta\eta} u$$

$$u\big|_{\eta=1} = 1,\ \ u\big|_{\eta=\infty} = 0,\ \ \phi' = -\left(\eta + \frac{1}{2\eta^2}\right)\cos\vartheta. \qquad [5]$$

The following standard initial conditions can be used:

$$u\big|_{t'=0} = 0,\ \ a'\big|_{t'=0} = 1\ \ (U'\big|_{t'=0} = 1). \qquad [6]$$

Because the left hand part of [4] is in the order of unity by the definition of t_*, we can define the characteristic value of the Nusselt number Nu_* by the following relation:

$$\epsilon^2 \mathrm{Ja}\mathrm{Nu}_* = 1. \qquad [7]$$

The number Nu∗ can also be found from the solution to problem [5]–[6] and, thus, it is the function of two parameters ϵ and St. Comparison of the non-stationary, convective (due to translatory motion) and conductive terms in [5] shows that

$$\mathrm{Nu}_*(\epsilon,\mathrm{St}) = \begin{cases} \epsilon^{-1}, & \mathrm{St} \gg 1 \\ \epsilon^{-1}, & \mathrm{St} \sim 1 \\ \epsilon^{-1}\mathrm{St}^{-1/2}, & \mathrm{St} \ll 1 \end{cases} \quad [8]$$

Note that any dimensionless constant number is a function of two basic numbers Ja and Pe_0. So, if we have a moderate thickness of the thermal boundary layer ($\mathrm{Nu}_* = 1$), then we find from [7] and [3] that

$$\mathrm{Nu}_* = 1: \ \epsilon = \mathrm{Ja}^{-1/2}, \ \mathrm{St} = 4\mathrm{Ja}\mathrm{Pe}_0^{-1}, \ t_* = a_0^2(\kappa_L \mathrm{Ja})^{-1}. \quad [9]$$

Evaluations [8] give the criterion when such values of the Nusselt number can be realized—both (Ja and Pe_0) numbers cannot high. Consequently, the regime with a thin thermal boundary layer ($\mathrm{Nu}_* \gg 1$) occurs when at least one of these numbers is high. Thus, in the case $\mathrm{St} \gg 1$ we have from [3], [7] and [8]:

$$\mathrm{Nu}_* = \mathrm{Ja} \gg 1: \ \epsilon = \mathrm{Ja}^{-1} \ll 1, \ \mathrm{St} = 4\mathrm{Ja}^2\mathrm{Pe}_0^{-1} = 4\delta^2 \gg 1, \ t_* = a_0^2(\kappa_L \mathrm{Ja}^2)^{-1}. \quad [10]$$

In the second limiting case $\mathrm{St} \ll 1$ we have

$$\mathrm{Nu}_* = \mathrm{Pe}_0^{1/2} \gg 1: \ \epsilon = \mathrm{Ja}^{-1/2}\mathrm{Pe}_0^{-1/4}, \ \mathrm{St} = 4\mathrm{Ja}\mathrm{Pe}_0^{-1/2} = 4\delta \ll 1, \ t_* = a_0^2(\kappa_L \mathrm{Ja}\mathrm{Pe}_0^{1/2})^{-1}. \quad [11]$$

Moderate values of the Strouhal number correspond to moderate values of the parameter δ. Note that in this case the requirement $\mathrm{Nu}_* \gg 1$ denotes $\epsilon \ll 1$.

Evaluations [10] and [11] show that the increasing Peclet and Jacob numbers cause a decreasing characteristic collapse time t_*. At very high Ja and Pe_0 it can be of the same order as the characteristic time of the empty sphere collapse t_i in the classical Rayleigh problem:

$$t_i = a_0(\rho_L/p_0)^{1/2}. \quad [12]$$

If $t_* < t_i$, then the vapor condenses so fast that the bubble radius cannot substantially change due to the liquid inertia. Thus, the pressure in the bubble drops, practically until zero, and the

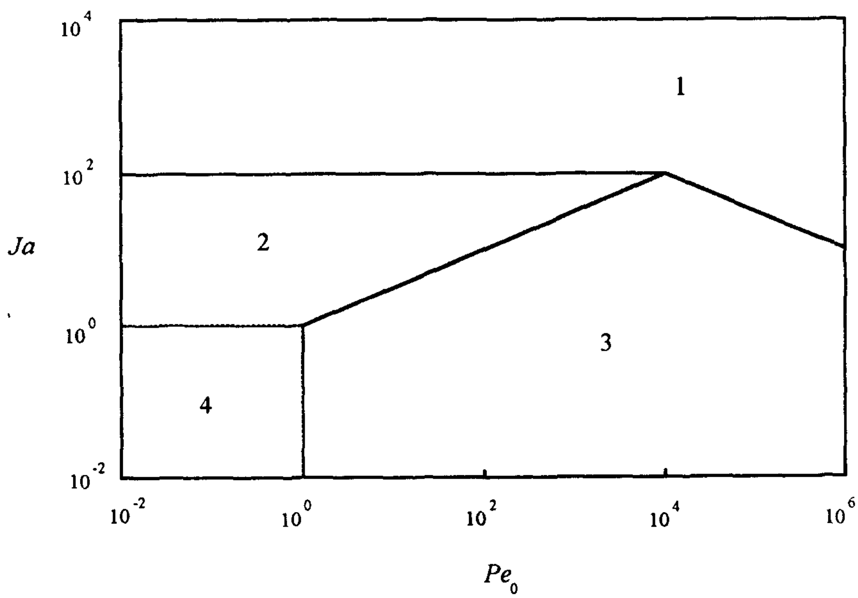

Figure 1. Areas in the space of parameters (Pe_0, Ja) with different characteristic times of the vapor bubble growth and collapse. Area 1 corresponds to the inertial regime of collapse with the characteristic time [12]. The thermal regime is realized for parameters from areas 2, 3 and 4. The characteristic times of the bubble growth and collapse in these areas can be determined by [10] for area 2, [11] for 3 and [9] for 4.

Rayleigh inertial regime of the bubble collapse takes place. The thermal regime can only take place if $t_* \gg t_i$. Consequently, the necessary conditions for this regime are the following:

$$\mathrm{Ja} \gg 1:\ \beta \mathrm{Ja}^2 \ll 1\ \ (\beta = \kappa_L a_0^{-1}(\rho_L/p_0)^{1/2}) \qquad [13]$$

$$\mathrm{Pe}_0 \gg 1:\ \beta \mathrm{Ja} \mathrm{Pe}_0^{1/2} \ll 1. \qquad [14]$$

In figure 1 the space of the parameters (Pe_0, Ja) is divided into four areas corresponding to various characteristic times of the collapse. The boundaries of areas 1–2 and 1–3 are determined by [13] and [14] and depend on the values of the parameter β. For the parameters from each area some simplifications of the general problem can be made. The present work deals with the parameters from areas 2 and 3. Note that a thin thermal boundary layer can be realized at the initial stage of the collapse.

4. ASYMPTOTIC EXPANSIONS AND NUSSELT NUMBER

The asymptotic analysis below is based on the assumption that $\mathrm{Nu}_* \gg 1$. This situation can be realized in a wide range of Strouhal numbers. So, we consider different asymptotic cases to provide full analysis of the regime with a thin thermal boundary layer.

4.1. *Moderate Strouhal numbers*

At moderate Strouhal numbers we have the small parameter $\epsilon \ll 1$. Let us introduce the boundary layer variable ζ and rewrite problem [5]–[6] in the zero-order approximation

$$(a')^2 \frac{\partial u}{\partial t'} - 3a' \frac{da'}{dt'} \zeta \frac{\partial u}{\partial \zeta} - \frac{6U'a'}{\mathrm{St}} \left[\zeta \cos\vartheta \frac{\partial u}{\partial \zeta} - \tfrac{1}{2} \sin\vartheta \frac{\partial u}{\partial \vartheta} \right] = \frac{\partial^2 u}{\partial \zeta^2}$$

$$u|_{\zeta=0} = 1,\ \ u|_{\zeta=\infty} = 0,\ \ u|_{t'=0} = 0\ \ (\zeta = \epsilon^{-1}(\eta - 1)). \qquad [15]$$

Note that the asymptotic of the boundary layer is formally justified at any angle except small vicinities of the critical points $\vartheta \sim \epsilon^2$ and $\pi - \vartheta \sim \epsilon^2$. Nevertheless, in zero-order approximation the contribution of these vicinities to the integral Nusselt number [3] is small (see Levich 1959; Gupalo *et al.* 1985).

If the following relationships hold

$$\xi = \zeta f(\vartheta, t'),\ \ v = g(\vartheta, t'),\ \ \tau = h(\vartheta, t'),\ \ \upsilon(\xi, v, \tau) = u(\zeta, \vartheta, t') \qquad [16]$$

$$f = (a')^3 \sin^2\vartheta,\ \ a' \frac{\partial g}{\partial t'} + \frac{3U'}{\mathrm{St}} \sin\vartheta \frac{\partial g}{\partial \vartheta} = 0,\ \ a' \frac{\partial h}{\partial t'} + \frac{3U'}{\mathrm{St}} \sin\vartheta \frac{\partial h}{\partial \vartheta} = (a')^5 \sin^4\vartheta \qquad [17]$$

then the spatial–temporal transform $(\zeta, \vartheta, t') \rightarrow (\xi, v, \tau)$ allows us to reduce problem [15] to the standard heat conduction problem:

$$\frac{\partial \upsilon}{\partial \tau} = \frac{\partial^2 \upsilon}{\partial \xi^2},\ \ \upsilon|_{\xi=0} = 1,\ \ \upsilon|_{\xi=\infty} = 0. \qquad [18]$$

For the initial condition $\tau_0 = h(\vartheta, 0) = 0$ we have $\upsilon|_{\tau=0} = 0$ and the self-similar solution of problem [18] takes place: $\upsilon = \mathrm{erfc}(\tfrac{1}{2}\xi\tau^{-1/2})$. So we can find the Nusselt number:

$$\mathrm{Nu} = -\frac{1}{\epsilon} \int_0^\pi f \frac{\partial \upsilon}{\partial \xi} |\xi = 0 \sin\vartheta d\vartheta = \frac{[a'(t')]^3}{\epsilon \pi^{1/2}} \int_0^\pi \frac{\sin^3\vartheta d\vartheta}{[h(\vartheta, t')]^{1/2}}. \qquad [19]$$

The integral representation for $h(\vartheta, t')$ can be found from the last equation [17]. In the variables

$$s(t') = 2 \int_0^{t'} \frac{U'(t'')}{a'(t'')} dt'',\ \ \gamma(\vartheta) = \tfrac{2}{3} \log[\tan(\tfrac{1}{2}\vartheta)],\ \ b(s) = \frac{(a')^5}{2U'},\ \ c(\gamma) = \sin^4\vartheta \qquad [20]$$

this equation can be rewritten in the form

$$\frac{\partial h}{\partial s} + \frac{1}{\mathrm{St}}\frac{\partial h}{\partial \gamma} = b(s)c(\gamma). \qquad [21]$$

The general solution of this first-order equation is

$$h = \int_0^s b(s')c(\gamma - \mathrm{St}^{-1}(s - s'))\,\mathrm{d}s' + H(\gamma - \mathrm{St}^{-1}s). \qquad [22]$$

An arbitrary function H is determined from the initial condition $\tau_0 = h(\vartheta,0) = 0$ and, as a result, $H \equiv 0$. Using [20] we can rewrite [22] in the form

$$h(\vartheta,t') = \int_0^{t'} \left\{a'(t'')\mathrm{sech}\left[\log\left(\tan\frac{\vartheta}{2}\right) - \frac{3}{\mathrm{St}}\int_{t'}^{t''}\frac{U'(t''')}{a'(t''')}\,\mathrm{d}t'''\right]\right\}^4 \mathrm{d}t''. \qquad [23]$$

Thus, the Nusselt number given by [19] and [23] depends on the bubble motion pre-history. Note that the case of moderate Strouhal numbers was considered by Ruckenstein & Davis (1971), who obtained similar expressions, but in a slightly different form. Expressions such as [19] and [23] are too unwieldy for practical use, and some simplifications are very desired much. These simplifications can be obtained immediately from [23] by asymptotic expansions at high and low Strouhal numbers. However, it is more convenient to consider [21].

4.2. High Strouhal numbers

To find the asymptotic expansion at $\mathrm{St} \gg 1$ we can represent the function h in the form

$$h = h_0(s,\gamma) + \mathrm{St}^{-1}h_1(s,\gamma) + \mathrm{St}^{-2}h_2(s,\gamma) + \ \ldots. \qquad [24]$$

Substituting [24] into [21] and collecting terms of the same order we can obtain

$$\frac{\partial h_0}{\partial s} = b(s)c(\gamma), \quad \frac{\partial h_{n+1}}{\partial s} = -\frac{\partial h_n}{\partial \gamma}, \quad h_n(0,\gamma) = 0, \quad n = 0,1,2, \ldots. \qquad [25]$$

Consequently, after integration we have

$$h_n = \alpha_n\left(-\frac{\mathrm{d}}{\mathrm{d}\gamma}\right)^n c, \quad \alpha_n(s) = \int_0^s \alpha_{n-1}(s')\,\mathrm{d}s', \quad \alpha_{-1} = b, \quad n = 0,1,2, \ldots. \qquad [26]$$

Using [20], [24] and [26] we find the first three terms of expansion h:

$$h = \sin^4\vartheta[\alpha_0 - 6\alpha_1\mathrm{St}^{-1}\cos\vartheta + 9\alpha_2\mathrm{St}^{-2}(5\cos^2\vartheta - 1) + \ \ldots]. \qquad [27]$$

From [19] and [27] we can find the expression for the Nusselt number:

$$\mathrm{Nu} = 2\epsilon^{-1}(\pi\alpha_0)^{-1/2}[a'(t')]^3\{1 + \tfrac{3}{2}(\alpha_0\mathrm{St})^{-2}(4\alpha_0\alpha_2 + 3\alpha_1^2) + O(\mathrm{St}^{-4})\}. \qquad [28]$$

According to [20] and [26] we have representations for α_n:

$$\alpha_0 = \int_0^{t'} [a'(t'')]^4\,\mathrm{d}t'', \quad \frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} = 2\frac{U'}{a'}\alpha_n, \quad \alpha_n(0) = 0, \quad n = 0,1,2, \ldots. \qquad [29]$$

The principal term of asymptotic [28], [29] was found by Plesset & Zwick (1954), who solved the problem in zero-order of approximation without translatory motion at $\mathrm{Ja} \gg 1$. The second term, expressing the effect of translatory motion is, presumably, a new result.

4.3. Low Strouhal numbers and moderate times

At $\mathrm{St} \ll 1$ the direct asymptotic expansion for h can be written in the form

$$h = \mathrm{St}h_1(s,\gamma) + \mathrm{St}^2h_2(s,\gamma) + \ \ldots. \qquad [30]$$

Using [21] and [30] we can obtain the following sequence of equations

$$\frac{\partial h_1}{\partial \gamma} = b(s)c(\gamma), \ \frac{\partial h_{n+1}}{\partial \gamma} = -\frac{\partial h_n}{\partial s}, \ h_n(s,-\infty) = 0, \ n=1,2, \ldots. \qquad [31]$$

The last boundary condition can be justified with the help of general integral representation [22]. Here we can substitute the integration variable with the relation $s' = s + \mathrm{St}(\gamma\text{-}\gamma')$ and assume that the Strouhal number tends to zero. Thus, solving the first two equations in [31] we have the following two-term asymptotic expansion of h

$$h = \mathrm{St}b(s)[\psi_1(\vartheta) - \mathrm{St}b_2(s)\psi_2(\vartheta) + \ldots]$$

$$\psi_1(\vartheta) = \int_{-\infty}^{\gamma} c(\gamma')\mathrm{d}\gamma' = \tfrac{4}{3}(\tfrac{2}{3} - \cos\vartheta + \tfrac{1}{3}\cos^3\vartheta), \ b_2(s) = \frac{1}{b}\frac{\mathrm{d}b}{\mathrm{d}s}$$

$$\psi_2(\vartheta) = \int_{-\infty}^{\gamma}\int_{-\infty}^{\gamma'} c(\gamma'')\,\mathrm{d}\gamma''\mathrm{d}\gamma' = -\tfrac{8}{27}\left(\log\left(\cos^2\frac{\vartheta}{2}\right) + \tfrac{1}{4}\sin^2\vartheta\right). \qquad [32]$$

Two first terms of the Nusselt number asymptotic can be found from [19] and [32]:

$$\mathrm{Nu} = \frac{[a'(t')]^3}{\epsilon[\pi \mathrm{St}b(s(t'))]^{1/2}}[I_1 + \tfrac{1}{2}I_2\mathrm{St}b_2(s(t')) + \ldots]$$

$$I_1 = 2\sqrt{2}, \ I_2 = \tfrac{4}{9}\sqrt{2}[2\sqrt{3} - 1 + 2\log(\sqrt{3} - 1)]\,. \qquad [33]$$

Using [3], [20] and [32], [33] can be rewritten in the form

$$\mathrm{Nu} = \frac{2}{\pi^{1/2}}\mathrm{Pe}_0^{1/2}(a'U')^{1/2}[1 + k_N\delta\frac{1}{(a')^4}\frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{(a')^5}{U'}\right) + O(\delta^2)], \ k_N = \tfrac{1}{4}I_2\sqrt{2} \approx 0.41. \qquad [34]$$

The principal term of this asymptotic depends only on the current Peclet number:

$$\mathrm{Nu} = 2\pi^{-1/2}\mathrm{Pe}^{1/2}. \qquad [35]$$

This formula is a well-known result for the quasi-stationary bubble Nusselt number at high Peclet and Reynolds numbers (Levich 1959). The second term of asymptotic [34] takes into account the effect of the non-stationarity of the thermal boundary layer (a new result). For example, if the velocity of the bubble is constant, then the increasing bubble radius ($\mathrm{d}a'/\mathrm{d}t' > 0$) leads to the decreasing thickness of the boundary layer due to its 'compression' by the moving boundary; in the case of bubble collapse ($\mathrm{d}a'/\mathrm{d}t' < 0$), increasing boundary layer occurs and the Nusselt number becomes less than its quasi-stationary value.

Note that the first two terms of the stationary Nusselt number asymptotic at high Reynolds numbers have the following form (see Gupalo *et al.* 1985)

$$\mathrm{Nu} = 2\pi^{-1/2}\mathrm{Pe}^{1/2}[1 - k_{\mathrm{R}}\mathrm{Re}^{-1/2}], \ k_{\mathrm{R}} = \tfrac{8}{5}(3\sqrt{3} - 2), \ \mathrm{Re} = 2aU/\nu_{\mathrm{L}} \qquad [36]$$

where ν_{L} is the viscosity of the liquid. Comparison of [34] and [36] shows that the second term in asymptotic [34] is more essential than the second term in [36] at

$$\mathrm{Ja} \gg \mathrm{Pr}^{1/2}, \ \mathrm{Pr} = \nu_{\mathrm{L}}/\kappa_{\mathrm{L}}. \qquad [37]$$

4.4. Low Strouhal numbers and small times

The direct expansion [30] is valid only at times $t' \gg \mathrm{St}$. At small times $t' \ll 1$ we introduce a new variable $\sigma = s/\mathrm{St}$. The integral representation of h [22] can be rewritten in the form

$$h = \mathrm{St}\int_0^{\sigma} b(\sigma'\mathrm{St})c(\gamma - \sigma + \sigma')\,\mathrm{d}\sigma'. \qquad [38]$$

According to [6] and [20] we have $b(\sigma'\mathrm{St}) = b(0) + O(\mathrm{St})$, $b(0) = \frac{1}{2}$. From this and [38] we find that the principal term of the asymptotic in the considered case takes the form

$$h = \tfrac{1}{2}\mathrm{St}\int_{\gamma-\sigma}^{\gamma} c(z)\,\mathrm{d}z, \quad z=\sigma' - \sigma + \gamma. \qquad [39]$$

Using [20] we can calculate integral [39] analytically and obtain the following expression

$$h = \tfrac{1}{9}\mathrm{St}\tau(1-\mu^2)^2(1+\mu\tau)^{-3}[3(1+\mu\tau) - \tau^2(1-\mu^2)]$$

$$\mu = -\tanh(\tfrac{3}{2}\gamma) = \cos\vartheta, \quad \tau = \tanh(\tfrac{3}{2}\sigma) \approx \tanh(3\mathrm{St}^{-1}t'). \qquad [40]$$

For the Nusselt number [19], consequently, we have ($a' = 1 + O(\mathrm{St})$)

$$\mathrm{Nu} = \frac{2\chi(\tau)}{(\pi\tau)^{1/2}}\mathrm{Pe}_0^{1/2}, \quad \chi(\tau) = \tfrac{3}{4}\int_{-1}^{1}\frac{(1+\mu\tau)^{3/2}\,\mathrm{d}\mu}{[3(1+\mu\tau) - \tau^2(1-\mu^2)]^{1/2}}. \qquad [41]$$

At $t'{\sim}1$ we have $\tau \approx 1$ (see [40]). From [41] it can be found that $\chi(0) = \frac{1}{2}\sqrt{3} \approx 0.86$, $\chi(1) = 1$ and [41] matches with [35] at moderate times. The limiting values of the monotonous function $\chi(\tau)$ are close to each other and sometimes for applications we can consider that $\chi \equiv \mathrm{const.} = 1$. A more accurate approximation of this function:

$$\chi(\tau) \approx \tfrac{1}{2}\sqrt{3}[1 + (\tfrac{2}{3}\sqrt{3} - 1)\tau^2] \qquad [42]$$

is convenient both for analytical and practical usage. The relative error of [42] is not greater than 2% at $0 \le \tau \le 1$.

5. BUBBLE COLLAPSE AT LOW STROUHAL NUMBERS

It was observed in the experimental studies of the collapse of vapor bubbles rising under gravity forces (Wittke & Chao 1967; Mayinger *et al.* 1991; see the next section) that the velocity of the bubble is practically constant. If we accept this empirical fact and consider that $U' \equiv 1$, then [4] and the expression for the Nusselt number obtained above form a closed system of equations. When [34] is justified, we can obtain the following equation describing the bubble dynamics

$$a'\frac{\mathrm{d}a'}{\mathrm{d}t'} = -\pi^{-1/2}(a')^{1/2}\left[1 + 5k_N\delta\frac{\mathrm{d}a'}{\mathrm{d}t'}\right]. \qquad [43]$$

After integrating this equation with the initial condition $a'(0) = 1$ we have

$$t' = \tfrac{2}{3}\pi^{1/2}[1 - (a')^{3/2}] + 5k_N\delta(1 - a'). \qquad [44]$$

The principal term of this asymptotic

$$t' = \tfrac{2}{3}\pi^{1/2}[1 - (a')^{3/2}] \qquad [45]$$

can be found directly using the quasi-stationary relationship [35] (see also Moalem & Sideman 1973).

As found in the previous section, [43] is not uniformly valid in the case of initial temperature drop. The uniformly valid zero-order approximation at $\mathrm{St} < 1$ can be obtained from the equation using [41]:

$$a'\frac{\mathrm{d}a'}{\mathrm{d}t'} = -(\pi\tau)^{-1/2}\chi(\tau)(a')^{1/2}, \quad a'(0) = 1. \qquad [46]$$

Taking into account [42] we can integrate this equation and find

$$1 - (a')^{3/2} = \delta F(\tau), \quad \tau = \tanh(\tfrac{3}{4}\delta^{-1}t')$$

$$F(\tau) = \pi^{-1/2}[(2\sqrt{3} - 4)\tau^{1/2} + \log(1 + \tau^{1/2}) - \log(1 - \tau^{1/2}) + 2\arctan(\tau^{1/2})]. \qquad [47]$$

The second term of the Nusselt number asymptotic [34] can be taken into account with the help of the asymptotic matching procedure for [46] and [43]. A uniformly valid combined asymptotic expansion in this case can be written in the form

$$1 - (a')^{3/2} + \tfrac{15}{2}\pi^{-1/2}k_N\delta(1 - a') = \delta F(\tau). \qquad [48]$$

Note, that some authors also derived their formulae for bubble dynamics with translatory motion using the assumption that $U' = (a')^n$, where the value of the exponent n depends on the regime of bubble rising in the presence of gravitation (see, for example, Moalem & Sideman 1973; Dimic 1977). It is not difficult to generalize [43]–[48] for this case by substituting $U' = (a')^n$ in [34].

6. COMPARISONS OF THE PRESENT RESULTS WITH PREVIOUS DATA

The work of Wittke & Chao (1967) is good for testing the present asymptotic results because some of the numerical studies in that work were carried out within the framework of the same equations describing the process. A constant velocity ($U' \equiv 1$) was assumed in the simulations. Also in the work of Wittke & Chao, all numerical results were presented using dimensionless time, $\tau_H = \frac{4}{\pi}Ja^2\kappa_L a_0^{-2}t$, the values of which are in the order of unity at high Strouhal numbers, but at St$\ll$1 we have $\tau_H = \frac{4}{\pi}\delta t'$ and the characteristic values of τ_H are small (t'~1, $\delta \ll 1$).

Good agreement between numerical (Wittke & Chao) and analytical results [45] on the bubble radius dynamics is shown in figure 2. Because in this case (Ja = 1, Pe_0~10^3) the values of the parameter δ are very small, we have the quasi-stationary regime of the heat transfer at all times of the bubble's life. Thus, being rebuilt in the new time scale, calculations for different Peclet numbers coincide with each other and with zero-order approximation [45].

The influence of the non-stationary nature of the thermal boundary layer on the bubble dynamics is seen in figure 3 (Ja = 10). The influence of the initial stage of the boundary layer development manifests itself in the S-shape of the condensation curves (for the curves of [44] and [45] we have $d^2a/dt^2 < 0$). Both curves calculated by [47] and [48] qualitatively agree with the curve obtained numerically. At a decreasing Peclet number (increasing δ) the simulated curve is closer to the curve corresponding to [48] than to the curve corresponding to [47].

Also in the work of Wittke & Chao (1967), their own experimental data were compared with the simulations and a satisfactory agreement was shown. However, a more complex situation was realized in the experiments and simulations than in the case considered in the present work, because

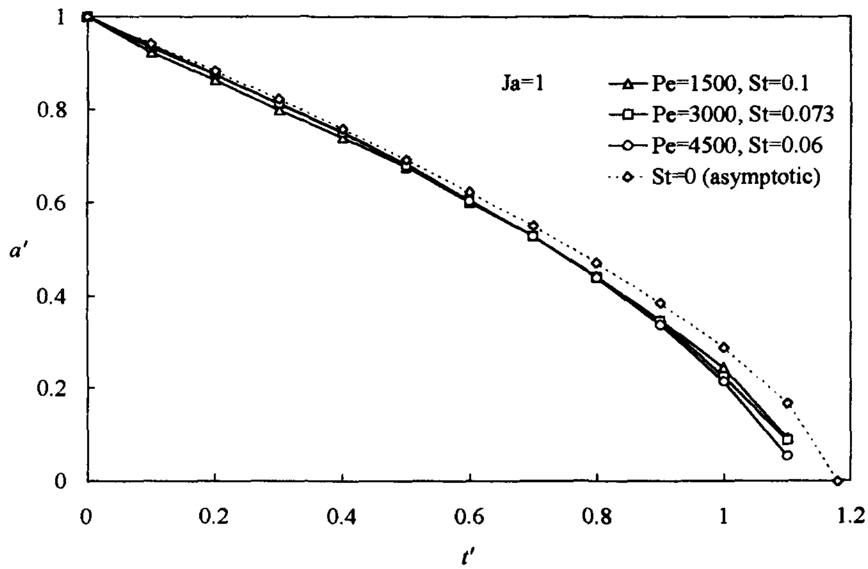

Figure 2. The dependence of the dimensionless bubble radius a' on dimensionless time t' at Ja = 1 and different Peclet numbers. The solid curves correspond to the numerical solutions of Wittke & Chao (1967). The dotted curve corresponds to the asymptotic formula [45] for the quasi-stationary heat transfer. The Strouhal numbers indicated on the chart are small in each case.

there were bubbles consisting of vapor and an inert gas, which is why the present results are not compared with these experiments. Nevertheless, the initial stage of the vapor–gas bubbles' dynamics in the case $\delta \ll 1$ can be satisfactorily described by [47] and [48].

The experimental data on pure vapor bubble collapse in subcooled liquids can be found in the works of Mayinger and collaborators (Nordmann & Mayinger 1981; Chen 1985; Mayinger *et al.* 1991). In contrast to the Wittke & Chao experiments, where condensation begins due to the instantaneous pressure drop, here vapor was injected into subcooled liquid through the nozzle. Vapor bubbles initially grew on the nozzle and then detached from it and collapsed in the bulk of the liquid. These authors also noticed, as did Wittke & Chao, that the rising velocity of the collapsing bubbles was close to constant.

The times of growth of the bubbles on the nozzle were of the same order of magnitude as the times of collapse. As well as the range of characteristic Jacob and Peclet numbers in the experiments which corresponded to $St \ll 1$, the quasi-stationary thermal boundary layer was built up on the bubbles practically at the moment of detachment. This conclusion can be confirmed by figure 4, where the data on the direct measurements of the Nusselt number and the graph of the quasi-stationary relationship [35] are plotted. The empirical approximation proposed by the experimenters, $Nu = 0.6Re_0^{0.6}Pr^{0.5}$ $(Re_0 = Pe_0/Pr)$, is also shown.

Thus, we can expect that bubble dynamics in the experiments of Mayinger and co-workers can be described by [44] and [45] if $Pe_0 \gg 1$ and $\delta \ll 1$. In figure 5 some comparisons of asymptotic results ([44] and [45]) with the experiments (Chen 1985) are given. Also the curves corresponding to the empirical correlation (Chen 1985), $a' = (1\text{-}0.56Re_0^{0.7}Pr^{0.5}JaFo)^{0.9}$, $Fo = \kappa_L t(2a_0)^{-2}$, which was obtained as a result of many his own experiments with various substances at $Ja = 5, \ldots, 80$, are plotted in figure 5.

These comparisons show that both [44] and [45] adequately describe the experimental data in the initial stages. However, because the radius of the bubble decreases, the influence of the non-stationarity of the thermal boundary layer increases. This peculiarity is clearly seen at sufficiently high Jacob numbers (increasing Ja leads to an increase of δ) and [44] much better describes the experiments than [45].

Also, to show the influence of parameter δ on the bubble dynamics, we have represented the original experimental data in the characteristic time scale $t' \sim 1$. Now it is clearly seen, that the value of this parameter effects bubble collapse.

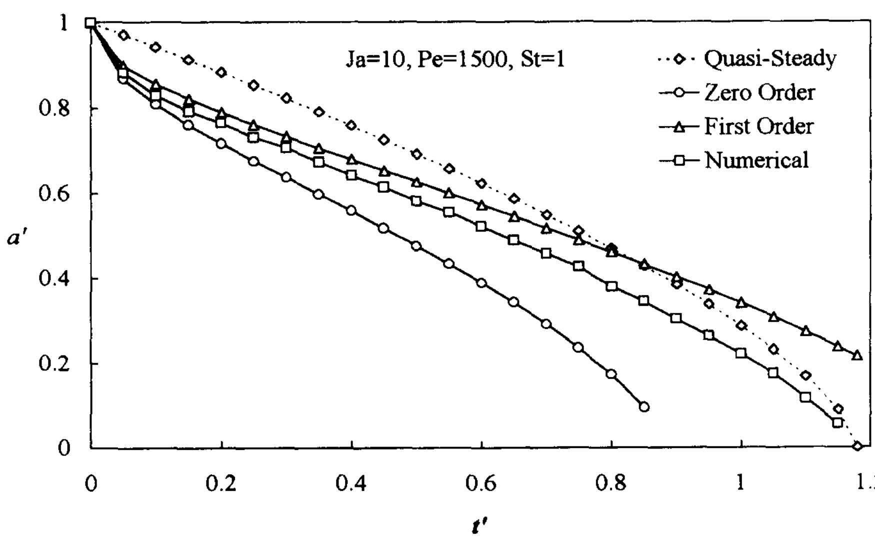

Figure 3. The dependence of the dimensionless bubble radius a' on dimensionless time t' at $Ja = 10$ and $Pe_0 = 1500$ ($St \approx 1.03$). The solid curves correspond to the asymptotic formulae [47] and [48] (zero and first orders of approximation, respectively) and the numerical simulations of Wittke & Chao (1967). The dotted curve corresponds to [45], when the non-uniformity of the quasi-steady solution at the initial stage is neglected.

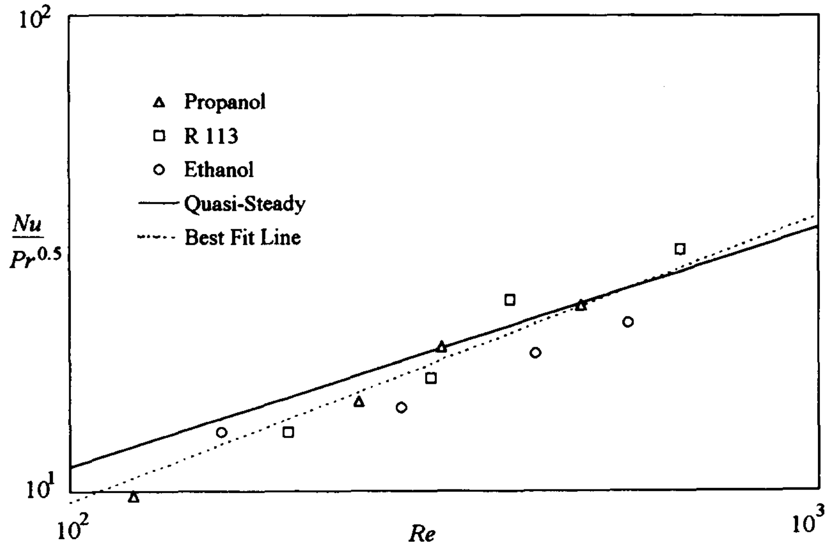

Figure 4. The dependence of the parameter $Nu/Pr^{1/2}$ on Reynolds number for a vapor bubble at the moment of detachment from the nozzle. The experimental data were obtained by Chen (1985) for various indicated media. The solid line shows the stationary dependence [35]. The dotted line represents the average experimental correlation (Chen 1985).

In figure 6 the generalized experimental data (Chen 1985) on the collapse times are plotted. The theoretical curves for collapse times predicted by [44] and [45] are plotted at $Pr = 10$ and $Pe_0 = 5.10^3$, $Pe_0 = 10^4$ (the characteristic values for these numbers), because the Peclet and Prandtl numbers vary from experiment to experiment. The difference between the theoretical curves corresponding to the indicated values of the Peclet number is small. Also there is little difference

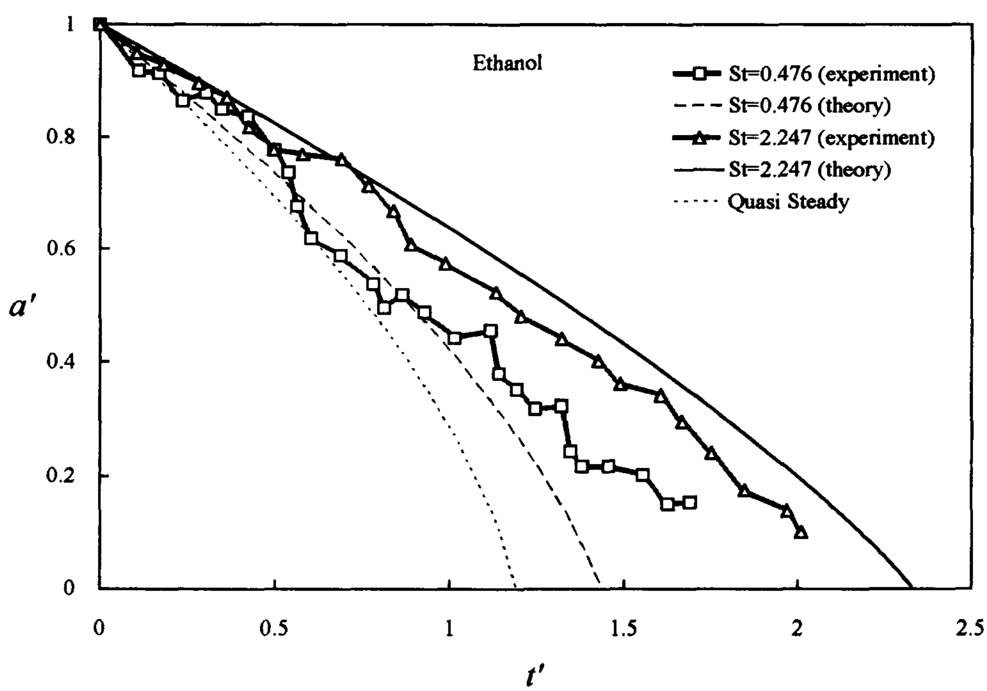

Figure 5. The influence of the Strouhal number on vapor bubble collapse in subcooled liquid. At the initial point a quasi-steady thermal boundary layer already exists on the bubble surface. The dashed ($\delta = \frac{1}{4}St = 0.119$) and solid ($\delta = 0.562$) curves are plotted using the asymptotic equation [44]. The dotted curve corresponds to the quasi-steady regime of heat transfer [45]. Experimental data were obtained by Chen (1985) for ethanol at a pressure in the system of 4 bars ($Ja = 9.57$, $Pr = 6.85$, $Pe_0 = 6466$, $St = 0.476$) and 1 bar ($Ja = 47.5$, $Pr = 11.4$, $Pe_0 = 7148$, $St = 2.247$).

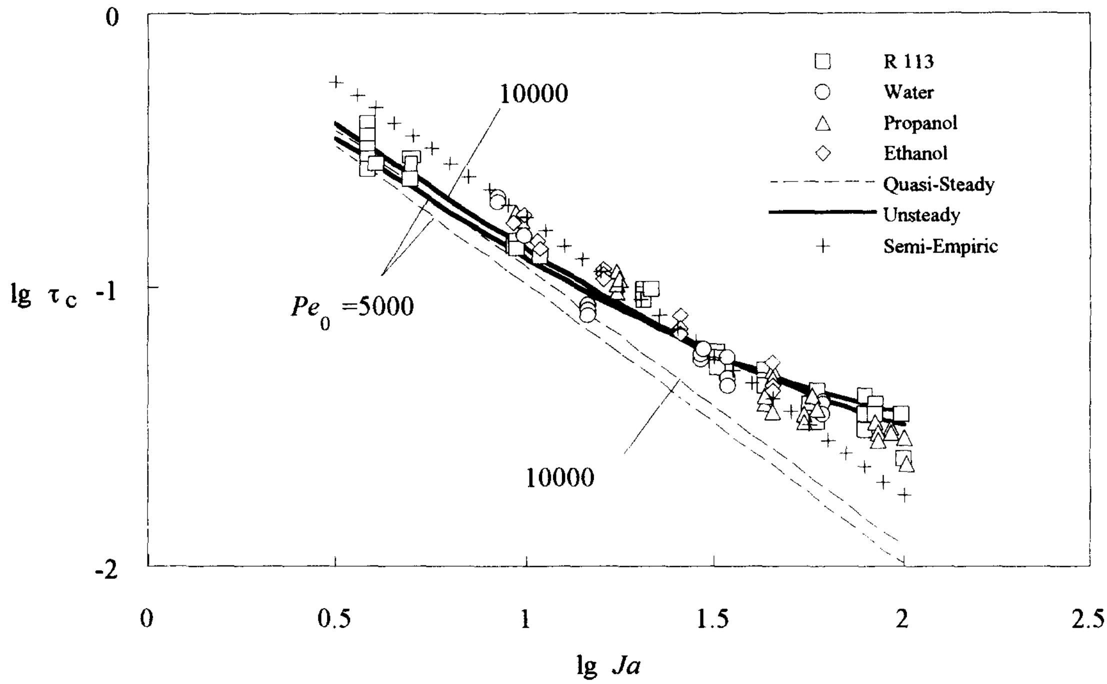

Figure 6. The dependence of the dimensionless time of the bubble collapse $\tau_c = \frac{\kappa_L t_c}{4a_0^2} Pe_0^{0.7} Pr^{0.2}$ (here t_c is the dimensional collapse time) on the Jacob number. The experiments with various liquids in various conditions were carried out by Chen (1985). The line indicated by crosses was obtained by Chen (1985) using the heat balance equation and his own averaged experimental data on the Nusselt number for rising and collapsing bubbles. The curves obtained within the framework of the present theory are plotted for Pr = 10 and two values of the Peclet number ($Pe_0 = 5000$ and $Pe_0 = 10{,}000$) that cover the range of Peclet numbers realized in Chen's experiments. The thick solid curves are calculated using the asymptotic equation [44] which takes into account the contribution of the non-stationary effects. The dotted lines corresponding to [45] show the dependence of heat transfer for the quasi-stationary regime.

between the curves with reasonable variations of the Prandtl number. Figure 6 shows that at $Ja < 10$ a theory, based on the assumption of the quasi-stationary character of the thermal boundary layer, describes the experiment well. Nevertheless, at sufficiently high Jacob numbers the effect of the increasing thickness of the boundary layer due to the quickly decreasing bubble radius can explain the experimental data qualitatively and quantitatively.

7. CONCLUSIONS

Analysis of the vapor bubble condensation curves based on the previous analytical and semi-empirical formulae of many authors (see Chen 1985; Mayinger *et al.* 1991) shows that there are no simple formulae $a(t)$ which quantitatively describe the experiments over a wide range of Jacob numbers and high Peclet numbers. Visualization of the processes taking place near the condensing bubbles (Nordmann & Mayinger 1981) displays a very complex flow near the bubbles, including strong deformations, the formation of jets, turbulence, etc. Nevertheless, the theory developed in the present work, based on the assumptions of the sphericity of the bubble and potential flow, is consistent with the experiments on averaged bubble radius dynamics and allows us to obtain simple analytical expressions for the non-stationary Nusselt number.

These expressions at various Jacob and Peclet numbers can be used not only for collapse prediction but also for description of the vapor bubble growth in superheated liquids. The effect of the non-stationarity of the boundary layer due to the changeability of the bubble velocity and radius can be strong and this effect can be taken into account in the complex models of multiphase flow. The obtained asymptotic of the Nusselt number at low Strouhal numbers, where the above-mentioned effect is represented by the differential term, is much more convenient than the general integral representation. Results obtained with the help of this asymptotic in a number of cases agree substantially better with the experiments and direct numerical simulations, than the results of the quasi-steady theory of heat and mass transfer.

Acknowledgements—The author wishes to thank Professor F. Mayinger (Munich, Germany) for kindly donating the experimental materials used partially in this work.

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